Boundary stabilization of elastodynamic systems. Part II: The case of a linear feedback.

Romain Brossard, Jean-Pierre Lohéac

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Abstract. In this paper, we extend some previous results published in [5].
We consider an elastodynamic system damped by a linear boundary feedback of Neumann-type. We prove stabilization results by using the multipliers method and Rellich-type relations given in the first part [6]. Especially, we take in account singularities which appear when changing boundary conditions.

Résumé. Nous généralisons ici les résultats publiés dans [5].
Nous considérons un système élastodynamique soumis à une rétroaction linéaire définie via une condition limite de type Neumann. Nous prouvons des résultats de stabilisation en utilisant la méthode des multiplicateurs et les relations de Rellich obtenues dans la première partie [6]. Nous prenons en compte les singularités qui apparaissent lorsque les conditions frontière changent.

Introduction

This paper is devoted to the study of boundary stabilization of the elastic wave equation. In this work, we extend the result obtained in [5] by using as a main tool results in [6].
Since we study a realistic geometrical case, singularities can be generated in the solution of our problem. Indeed these can appear when the feedback is defined on a strict subset of the boundary, especially when the boundary is connected.
We have chosen to study our problem by the multipliers method. This method is not optimal: its leads only to sufficient conditions, but its main advantage lies in the fact that it leads to explicit decay rates of the energy function.
Similar problems have been addressed by many authors. We especially mention:

- the case of the waves equation (see [11] and the references therein). Geometrical assumptions involving singularities have been firstly studied in [7]. This work has been later extended in [13] and [4].
- the case of elastodynamic systems:

$$\begin{cases} \mathbf{u}'' - \operatorname{div}(\sigma(\mathbf{u})) = \mathbf{0}\,, & \text{in } \Omega \times \mathbb{R}_+\,, \\ \mathbf{u} = \mathbf{0}\,, & \text{on } \partial\Omega_D \times \mathbb{R}_+\,, \\ \sigma(\mathbf{u})\boldsymbol{\nu} = \mathbf{F}(\mathbf{x}, \mathbf{u}, \mathbf{u}')\,, & \text{on } \partial\Omega_N \times \mathbb{R}_+\,, \\ \mathbf{u}(0) = \mathbf{u}_0\,, & \text{in } \Omega\,, \\ \mathbf{u}'(0) = \mathbf{u}_1\,, & \text{in } \Omega\,. \end{cases} \tag{1}$$

In [14], Lagnese has introduced a “natural” feedback: $\mathbf{F}(\mathbf{x}, \mathbf{u}, \mathbf{u}') = -a\mathbf{u} - b\mathbf{u}'$. In [15, 12], another feedback is introduced, in order to obtain a stabilization result. In [1], Alabau and Komornik obtained a stabilization result with the natural feedback. This work has been extended in [8, 9, 2, 3]. In all these works, geometrical restrictions are such that singularities do not appear. The most

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general result concerning Lamé system has been obtained by M. A. Horn [10], using micro-local analysis techniques.

Here, as well as in the case of waves equation, our generalized Rellich relations obtained in [6] leads us to consider a particular feedback law, inspired by the case of scalar wave equation, and to give boundary stabilization results for the elastodynamic system under weaker geometrical restrictions.
The case of the general natural feedback when singularities appear is still open.
Let us now introduce notations and main assumptions.
We here consider an elastic body which satisfies Lamé's laws. As usual, we define the strain tensor and the stress tensor for a regular vector field $\mathbf{v}$ by

$$\epsilon_{ij}(\mathbf{v}) = \frac{1}{2}\left(\partial_i \mathbf{v}_j + \partial_j \mathbf{v}_i\right), \qquad \sigma(\mathbf{v}) = 2\mu\,\epsilon(\mathbf{v}) + \lambda\,\mathrm{div}(\mathbf{v}) I_n \ ,,$$

where λ and μ are the Lamé's coefficients and I_n is the identity matrix of $\mathbb{R}^n$.
Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set such that its boundary satisfies

$$\partial\Omega = \partial\Omega_D \cup \partial\Omega_N\,, \quad \text{with} \left\{ \begin{array}{l} \partial\Omega_D \cap \partial\Omega_N = \emptyset\,, \\ \mathrm{meas}_{\partial\Omega}(\partial\Omega_D) \neq 0\,, \\ \mathrm{meas}_{\partial\Omega}(\partial\Omega_N) \neq 0\,. \end{array} \right. \tag{2}$$

We denote the boundary interface by

$$\Gamma = \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N}\,.$$

We assume that Ω is smooth enough so that for almost every $\mathbf{x} \in \partial\Omega$, we can consider $\boldsymbol{\nu}(\mathbf{x})$ the normal unit vector pointing outward of Ω.
We assume moreover that there exists $\mathbf{x}_0 \in \mathbb{R}^n$ such that, setting $\mathbf{m}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$, we have (see Figures 1 and 2)

$$\mathbf{m}\cdot\boldsymbol{\nu} \leq 0\,, \ \text{ on } \partial\Omega_D\,, \qquad \mathbf{m}\cdot\boldsymbol{\nu} \geq 0\,, \ \text{ on } \partial\Omega_N\,, \qquad \mathrm{meas}_{\partial\Omega}\{\mathbf{x} \in \partial\Omega_N \,/\, \mathbf{m}(\mathbf{x})\cdot\boldsymbol{\nu}(\mathbf{x}) > 0\} \neq 0\,. \tag{3}$$

It can be observed that, if $\Gamma \neq \emptyset$ and if $\partial\Omega$ is smooth enough, then

$$\mathbf{m}\cdot\boldsymbol{\nu} = 0\,, \quad \text{on } \Gamma\,. \tag{4}$$

We here consider the linear isotropic elastodynamic system

$$\left\{ \begin{array}{ll} \mathbf{u}'' - \mathrm{div}(\sigma(\mathbf{u})) = \mathbf{0}\,, & \text{in } \Omega \times \mathbb{R}_+\,, \\ \mathbf{u} = \mathbf{0}\,, & \text{on } \partial\Omega_D \times \mathbb{R}_+\,, \\ \sigma(\mathbf{u})\boldsymbol{\nu} = -(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}'\,, & \text{on } \partial\Omega_N \times \mathbb{R}_+\,, \\ \mathbf{u}(0) = \mathbf{u}_0\,, & \text{in } \Omega\,, \\ \mathbf{u}'(0) = \mathbf{u}_1\,, & \text{in } \Omega\,. \end{array} \right. \tag{5}$$

As well as in the case of waves equation, because of the change of boundary conditions, singularities appear along Γ. Nevertheless, we have seen in [6] that, since the feedback vanishes on Γ (cf (4)), we can obtain Rellich-type relations.
We introduce following Sobolev spaces: $\mathbb{L}^2(\Omega) = (\mathrm{L}^2(\Omega))^n$, $\mathbb{H}^s(\Omega) = (\mathrm{H}^s(\Omega))^n$, for some $s > 0$, and $\mathbb{H}^1_D(\Omega) = \{\mathbf{v} \in \mathbb{H}^1(\Omega) \,/\, \mathbf{v} = \mathbf{0}\,, \ \text{on } \partial\Omega_D\}$.

Well-posedness

The proof of well-posedness of problem (5) is left to the reader. It can be done by using the semi-group method.
One first can built strong solutions after introducing the operator $\mathcal{A}$ such that

$$\begin{array}{l} D(\mathcal{A}) = \{(\mathbf{u},\mathbf{v}) \in \mathbb{H}^1_D(\Omega) \times \mathbb{H}^1_D(\Omega) \,/\, \mathrm{div}(\sigma(\mathbf{u})) \in \mathbb{L}^2(\Omega) \ \text{and} \ \sigma(\mathbf{u})\boldsymbol{\nu} = -(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{v}\,, \text{on } \partial\Omega_N\}\,, \\ \mathcal{A}(\mathbf{u},\mathbf{v}) = (\mathbf{v}, \mathrm{div}(\sigma(\mathbf{u}))\,. \end{array} \tag{6}$$

The second step consists in proving the existence of weak solutions by a density argument, provided that initial data satisfy

$$(\mathbf{u}_0, \mathbf{u}_1) \in \mathbb{H}^1_D(\Omega) \times \mathbb{L}^2(\Omega)\,. \tag{7}$$

The uniqueness is obtained thanks to the linearity of the problem.

Energy function

Energy function is given by

$$E(\mathbf{u},t) = \frac{1}{2}\int_\Omega \left(|\mathbf{u}'|^2 + \sigma(\mathbf{u}):\epsilon(\mathbf{u})\right) d\mathbf{x}\,. \tag{8}$$

Under above assumptions, by applying Green's formula, one can easily prove that the time-derivative of the energy function satisfies for almost every $t > 0$,

$$E'(\mathbf{u},t) = -\int_{\partial\Omega_N} \mathbf{m}\cdot\boldsymbol{\nu}\,|\mathbf{u}'|^2\,d\gamma\,. \tag{9}$$

Then the energy function is decreasing with respect to time.

Stabilization result

Furthermore, we here prove, under convenient geometric assumptions, results of uniform exponential stabilization in the following form

There exist two constants $C > 0$ and $\varpi > 0$ such that for all $(\mathbf{u}_0, \mathbf{u}_1)$ satisfying (7)*, the solution $\mathbf{u}$ of* (5) *satisfies*

$$\forall t \in \mathbb{R}_+\,,\quad E(\mathbf{u},t) \le C\,E(\mathbf{u},0)\,\exp(-\varpi t)\,.$$

This paper is organized as follows.

- In Section 1, we recall the boundary stabilization result obtained in [5] when Ω is a bi-dimensional convex polygonal domain (Theorem 1.1).
- In Section 2, we give the extension of this result when Ω is a n-dimensional smooth domain (Theorem 2.1).
- In Section 3, we prove Theorem 2.1.

1 Case of a bi-dimensional convex polygonal domain

We here express the result in the case of a convex polygonal domain $\Omega \subset \mathbb{R}^2$. We assume that its boundary $\partial\Omega$ satisfies (2) and furthermore,

$$\Gamma = \{\mathbf{s}_1, \mathbf{s}_2\}\,, \tag{10}$$

where $\mathbf{s}_1$ and $\mathbf{s}_2$ will be considered as two vertices of $\partial\Omega$ (see Figure 1).

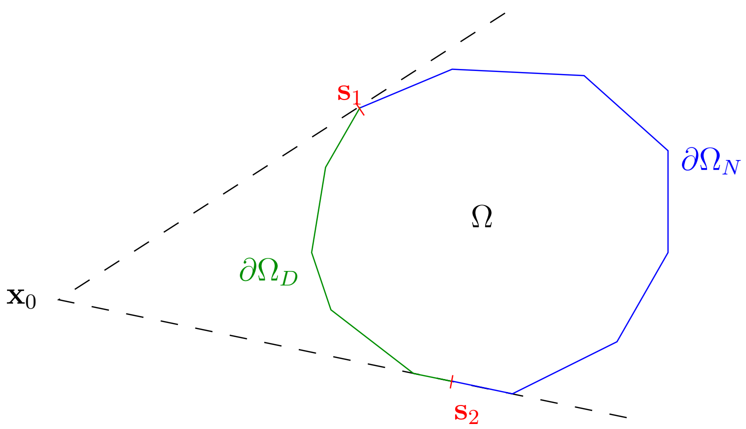

Figure 1: a convex polygonal domain Ω with a non-empty interface $\Gamma = \{\mathbf{s}_1, \mathbf{s}_2\}$.

Among points of interface Γ, we consider the set Γ_π of points where edges gives an angle π. At such points, we can define a tangent vector $\boldsymbol{\tau}$ pointing from $\partial\Omega_N$ to $\partial\Omega_D$ (see Figure 1).

The following Theorem has been announced in [5]. Its proof is similar to the proof of Theorem 2.1 given in section 3.

Theorem 1.1 — *Let Ω be a convex polygonal domain of $\mathbb{R}^2$ satisfying (2), (3) and (10).*
There exist $C > 0$ and $\varpi > 0$ such that for every $(\mathbf{u}_0, \mathbf{u}_1)$ satisfying (7), the solution $\mathbf{u}$ of (5) satisfies

$$\forall t \in \mathbb{R}_+ , \quad E(\mathbf{u}, t) \leq C\, E(\mathbf{u}, 0) \exp(-\varpi t) .$$

2 Case of a n-dimensional smooth domain

We consider a bounded connected domain $\Omega \subset \mathbb{R}^n$. We assume that its boundary $\partial\Omega$ is of class $\mathcal{C}^2$ and satisfies (2). Furthermore we assume (see Figure 2)

$$\begin{array}{l} \Gamma \text{ is a } (n-2)\text{-dimensional submanifold of class } \mathcal{C}^3 \\ \text{there exists a neighborhood } \Omega' \text{ of } \Gamma \text{ such that } \partial\Omega \cap \Omega' \text{ is a } (n-1)\text{-submanifold of class } \mathcal{C}^3 \end{array} \tag{11}$$

We assume that there exists $\mathbf{x}_0 \in \mathbb{R}^n$ such that (3) is satisfied. At each point $\mathbf{s}$ of Γ, we consider Γ as a submanifold of $\partial\Omega$ of co-dimension 1 and we can denote by $\boldsymbol{\tau}(\mathbf{s})$ the unit normal vector to Γ pointing outward of $\partial\Omega_N$. Let us write here our main assumption (see Figure 2)

$$\mathbf{m}\cdot\boldsymbol{\tau} \leq 0 , \quad \text{on } \Gamma. \tag{12}$$

We emphasize that such a condition appears in the case of waves equation [4]. When using multipliers method, we shall take in account this condition in Rellich-type relation and this will give the behavior of the energy function.
Condition (12) is especially satisfied when Ω is convex, and $\mathbf{x}_0$ outside of $\overline{\Omega}$.

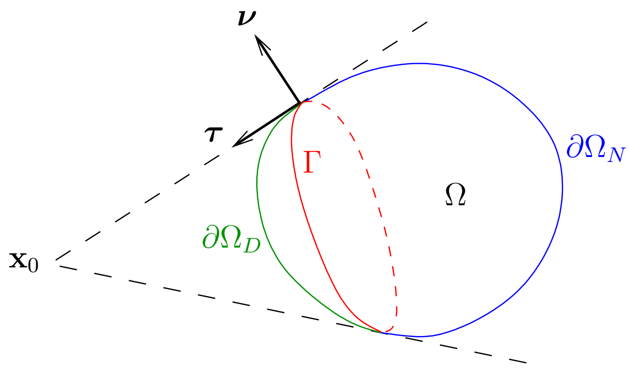

Figure 2: a general smooth domain Ω with a non-empty interface.

Theorem 2.1 — *Let Ω be a bounded connected domain of $\mathbb{R}^n$. We assume that its boundary $\partial\Omega$ is of class $\mathcal{C}^2$ and satisfies (2) and (11). We assume moreover that there exists $\mathbf{x}_0 \in \mathbb{R}^n$ such that (3) and (12) hold.*
Then there exist $C > 0$ and $\varpi > 0$ such that for every $(\mathbf{u}_0, \mathbf{u}_1)$ satisfying (7), the solution $\mathbf{u}$ of (5) satisfies

$$\forall t \in \mathbb{R}_+ , \quad E(\mathbf{u}, t) \leq C\, E(\mathbf{u}, 0) \exp(-\varpi t) .$$

In order to prove Theorem 2.1, we follow the method used in [13] and described in [11], so-called the multipliers method.

Some notations

For a regular vector field $\mathbf{u}$, we define

$$\nabla \mathbf{u} = (\partial_j u_i)_{1\le i,j\le n} = \begin{pmatrix} \partial_1 u_1 & \dots & \partial_n u_1 \\ \vdots & \ddots & \vdots \\ \partial_1 u_n & \dots & \partial_n u_n \end{pmatrix}.$$

As well as in [6], for two regular vector fields $\mathbf{v}_1$ and $\mathbf{v}_2$, we define

$$\Theta(\mathbf{v}_1, \mathbf{v}_2) = 2\,(\sigma(\mathbf{v}_1)\boldsymbol{\nu})\cdot((\mathbf{m}\cdot\nabla)\mathbf{v}_2) - (\mathbf{m}\cdot\boldsymbol{\nu})\,(\sigma(\mathbf{v}_1):\epsilon(\mathbf{v}_2))\,.$$

We first prove Theorem 2.1 for strong solutions. The result for weak solutions follows thanks to a density result.

Main tools

Our first main tool is a Rellich-type relation obtained in [6] (Theorem 4.1), which we adapt here in the following Proposition.

Proposition 2.1 — *Let $\Omega \subset \mathbb{R}^n$ ($n \ge 3$) be a bounded connected domain of class $\mathcal{C}^2$ which satisfies* (2)-(4). *Let $\mathbf{u} \in \mathbb{H}^1(\Omega)$ such that*

$$\mathrm{div}(\sigma(\mathbf{u})) \in \mathbb{L}^2(\Omega)\,, \quad \mathbf{u} \in \mathbb{H}^{3/2}(\partial\Omega_D)\,, \quad \sigma(\mathbf{u})\cdot\boldsymbol{\nu} \in \mathbb{H}^{1/2}(\partial\Omega_N)\,. \tag{13}$$

Then $\Theta(\mathbf{u},\mathbf{u})$ belongs to $\mathbb{L}^1(\partial\Omega)$ and there exists $\zeta \in \mathrm{L}^2(\Gamma)$, depending on local singularity coefficients of $\mathbf{u}$, such that

$$2\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot((\mathbf{m}\cdot\nabla)\mathbf{u})\,d\mathbf{x} = (n-2)\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,d\mathbf{x} + \int_{\partial\Omega}\Theta(\mathbf{u},\mathbf{u})\,d\boldsymbol{\gamma} + \int_\Gamma |\zeta|^2\,\mathbf{m}\cdot\boldsymbol{\tau}\,d\mathbf{s}\,.$$

for $n = 2$, see Theorems 2.1 and 3.1 in [6].

The second main tool is a Gronwall-type formula. Its proof can be found in [11], for example.

Proposition 2.2 — *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function such that there exists $C > 0$ independent of t such that*

$$\int_t^\infty E(s)\,ds \le C\,E(t)\,, \quad \forall t \ge 0\,, \tag{14}$$

then we have

$$E(t) \le E(0)\exp\Big(1 - \frac{t}{C}\Big)\,, \quad \forall t \ge C\,.$$

3 Proof of Theorem 2.1

Our proof is composed of three main steps.

First step

Let us consider a strong solution $\mathbf{u}$ of problem (5).
We can easily prove that the energy function is non-increasing with respect to time by using (9).

Second step

We now use the multipliers method (see for example [11] and [17]) to prove that the energy function satisfies (14).
Let $T > S > 0$ be two constants. We now introduce $M\mathbf{u} = 2(\mathbf{m}\cdot\nabla)\mathbf{u} + (n-1)\mathbf{u}$. We have

$$\int_S^T \int_\Omega \mathbf{u}''\cdot M\mathbf{u}\,d\mathbf{x}\;dt = \int_S^T\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot M\mathbf{u}\,d\mathbf{x}\,dt\,. \tag{15}$$

Integrating the left-hand side by parts with respect to t, we get

$$\int_S^T \int_\Omega \mathbf{u}''\cdot M\mathbf{u}\,d\mathbf{x}\,dt = \Big[\int_\Omega \mathbf{u}'\cdot M\mathbf{u}\,d\mathbf{x}\Big]_S^T - \int_S^T \int_\Omega \mathbf{u}'\cdot M\mathbf{u}'\,d\mathbf{x}\,dt\,.$$

But we can write:

$$\begin{aligned} 2\int_\Omega \mathbf{u}'\cdot((\mathbf{m}\cdot\nabla)\mathbf{u}')\,d\mathbf{x} &= \int_\Omega (\mathbf{m}\cdot\nabla)|\mathbf{u}'|^2\,d\mathbf{x} \\ &= \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{u}'|^2\,d\gamma - \int_\Omega \mathrm{div}(\mathbf{m})|\mathbf{u}'|^2\,d\mathbf{x} \\ &= \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{u}'|^2\,d\gamma - n\int_\Omega |\mathbf{u}'|^2\,d\mathbf{x}. \end{aligned}$$

Therefore,

$$\int_S^T \int_\Omega \mathbf{u}''\cdot M\mathbf{u}\,d\mathbf{x}\,dt = \Big[\int_\Omega \mathbf{u}'\cdot M\mathbf{u}\,d\mathbf{x}\Big]_S^T + \int_S^T \int_\Omega |\mathbf{u}'|^2\,d\mathbf{x}\,dt - \int_S^T \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{u}'|^2\,d\gamma\,dt\,. \tag{16}$$

The right-hand side of (15) can be written as follows,

$$\int_S^T \int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot M\mathbf{u}\,d\mathbf{x}\,dt = 2\int_S^T \int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot((\mathbf{m}\cdot\nabla)\mathbf{u})\,d\mathbf{x}\,dt + (n-1)\int_S^T \int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot\mathbf{u}\,d\mathbf{x}\,dt\,.$$

Green's formula gives

$$\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot\mathbf{u}\,d\mathbf{x}\,dt = -\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})(\mathbf{u}'\cdot\mathbf{u})\,d\gamma\,dt - \int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,d\mathbf{x}\,dt\,.$$

Now, since $\mathbf{u}$ is a strong solution of (5), we can observe that for each time t, $(\mathbf{u}(t),\mathbf{u}'(t))$ belongs to $D(\mathcal{A})$ (see (6)). Hence, $\mathbf{u}(t)$ satisfies conditions of Proposition 2.1. Thanks to (12), the integral on Γ is non-positive and we get

$$2\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot((\mathbf{m}\cdot\nabla)\mathbf{u})\,d\mathbf{x} \le (n-2)\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,d\mathbf{x} + \int_{\partial\Omega} \Theta(\mathbf{u},\mathbf{u})\,d\gamma\,.$$

We now take in account the boundary conditions in problem (5).
Especially, since we have $\mathbf{u}=\mathbf{0}$ on $\partial\Omega_D$, we get for all i, $\nabla u_i = (\nabla u_i\cdot\boldsymbol{\nu})\boldsymbol{\nu}$ on $\partial\Omega_D$.
Hence, $(\sigma(\mathbf{u})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{u} = (\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\mathbf{u}):\epsilon(\mathbf{u})$ on $\partial\Omega_D$.
Then, we can rewrite the second term of above right-hand side as follows,

$$\int_{\partial\Omega} \Theta(\mathbf{u},\mathbf{u})\,d\gamma = -\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\left[2\mathbf{u}'\cdot((\mathbf{m}\cdot\nabla)\mathbf{u}) + \sigma(\mathbf{u}):\epsilon(\mathbf{u})\right]d\gamma + \int_{\partial\Omega_D} (\mathbf{m}\cdot\boldsymbol{\nu})\,(\sigma(\mathbf{u}):\epsilon(\mathbf{u}))\,d\gamma\,.$$

Thanks to (3), we get

$$\int_{\partial\Omega} \Theta(\mathbf{u},\mathbf{u})\,d\gamma \le -\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\left[2\mathbf{u}'\cdot(\mathbf{m}\cdot\nabla)\mathbf{u} + \sigma(\mathbf{u}):\epsilon(\mathbf{u})\right]d\gamma\,,$$

and

$$2\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,d\mathbf{x} \le (n-2)\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,d\mathbf{x} - \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\left[2\mathbf{u}'\cdot((\mathbf{m}\cdot\nabla)\mathbf{u}) + \sigma(\mathbf{u}):\epsilon(\mathbf{u})\right]d\gamma\,.$$

Finally, the right-hand side of (15) can be bounded as follows,

$$\begin{aligned} \int_S^T \int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot M\mathbf{u}\,d\mathbf{x}\,dt \quad\le\quad & -\int_S^T \int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,d\mathbf{x}\,dt \\ & -\int_S^T \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\left[2\mathbf{u}'\cdot((\mathbf{m}\cdot\nabla)\mathbf{u}) + \sigma(\mathbf{u}):\epsilon(\mathbf{u})\right]d\gamma\,dt. \\ & -(n-1)\int_S^T \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\,(\mathbf{u}'\cdot\mathbf{u})\,d\gamma\,dt\,. \end{aligned} \tag{17}$$

From (15), (16) and (17), we deduce

$$
\begin{aligned}
2\int_S^T E(\mathbf{u},t)\,dt \quad \leq \quad & -\Big[\int_\Omega \mathbf{u}'\cdot M\mathbf{u}\,d\mathbf{x}\Big]_S^T + \int_S^T \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{u}'|^2\,d\gamma\,dt \\
& -(n-1)\int_S^T \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\,(\mathbf{u}'\cdot\mathbf{u})\,d\gamma\,dt \\
& -\int_S^T \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\,\big[2\mathbf{u}'\cdot((\mathbf{m}\cdot\nabla)\mathbf{u}) + \sigma(\mathbf{u}):\epsilon(\mathbf{u})\big]\,d\gamma\,dt\,.
\end{aligned}
\tag{18}
$$

One can easily prove that there exists $C_1 > 0$ such that for all $t > 0$,

$$
\Big|\int_\Omega \mathbf{u}'\cdot M\mathbf{u}\,d\mathbf{x}\Big| \leq C_1\,E(\mathbf{u},t)\,.
$$

Then, since the energy function is non-increasing, we get

$$
\Big|\Big[\int_\Omega \mathbf{u}'\cdot M\mathbf{u}\,d\mathbf{x}\Big]_S^T\Big| \leq 2\,C_1\,E(\mathbf{u},S)\,. \tag{19}
$$

Moreover, for every $\theta > 0$, we get $C_2 > 0$ independent of S and T such that

$$
\Big|\int_S^T \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\,\mathbf{u}'\cdot\mathbf{u}\,d\gamma\,dt\Big| \leq \theta\int_S^T E(\mathbf{u},t)\,dt + C_2\,E(\mathbf{u},S)\,. \tag{20}
$$

At last, we use the following lemma which is proved at the end of this paper.

Lemma 3.1 *For all $\theta > 0$ small enough, there exists $C_\theta > 0$ independent of $\mathbf{u}$, S and T such that*

$$
-\int_S^T \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\,\big[2\mathbf{u}'\cdot((\mathbf{m}\cdot\nabla)\mathbf{u}) + \sigma(\mathbf{u}):\epsilon(\mathbf{u})\big]\,d\gamma dt \leq \theta\int_S^T E(\mathbf{u},t)\,dt + C_\theta\,E(\mathbf{u},S)\,. \tag{21}
$$

Hence, from (18), (19), (20) and (21), we obtain that for every $\theta > 0$ small enough, there exists $C_3 > 0$ independent of S and T such that

$$
(2-\theta)\int_S^T E(\mathbf{u},t)\,dt \leq C_3\,E(\mathbf{u},S)\,.
$$

It is convenient to take θ small enough (at least $\theta < 2$). The limit when T tends to infinity gives the required result and we may apply Proposition 2.2.

Third step

This result can be extended to weak solutions by a density argument, since decay rate does not depend on the considered strong solution. □

Remark. All constants that appear in the above proof are explicit, so we can get the exponential decay rate with respect to geometrical data.

Proof of Lemma 3.1

For the proof of this Lemma, we follow [3]. In particular, we use local coordinates in order to estimate some boundary integral terms.

In the following lines, θ is some arbitrary positive constant and we will denote by C a generic constant independent of $\mathbf{u}$ and t.

Observe that we will often use Landau's notations ($O(.)$) with similar conventions.

Since $\partial\Omega$ is of class $\mathcal{C}^2$, for all $\mathbf{x} \in \partial\Omega$, we can build a local $\mathcal{C}^2$-diffeomorphism $\boldsymbol{\phi}$ from an open subset $V_{\mathbf{x}} \subset \mathbb{R}^{n-1}$ onto an open neighborhood $\mathcal{V}_{\mathbf{x}} \subset \partial\Omega$. Then vectors

$$
\mathbf{a}_i(\mathbf{x}) = \frac{\partial\boldsymbol{\phi}}{\partial\xi_i}(\boldsymbol{\phi}^{-1}(\mathbf{x}))\,, \quad \forall i \in \{1,\ldots,n-1\}\,,
$$

are independent and generate $T_{\mathbf{x}}(\partial\Omega)$, the tangent space at point $\mathbf{x}$. Moreover, we denote by $T(\partial\Omega)$ the tangent bundle (see [16] and [19]).
We then denote by g the metric tensor related to $\boldsymbol{\phi}$: $\quad g_{ij} = \mathbf{a}_i\cdot\mathbf{a}_j\,,\quad \forall(i,j)\in\{1,\dots,n-1\}^2$,
and by $(g^{ij})_{(i,j)\in\{1,\dots,n-1\}^2}$ its inverse.
We denote by $\pi(\mathbf{x})$ the orthogonal projection on $T_{\mathbf{x}}(\partial\Omega)$. Then, for every vector field $\mathbf{v}$ defined on $\overline{\Omega}$, we have for almost every $\mathbf{x}$ in $\partial\Omega$,

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}_T(\mathbf{x}) + v_\nu(\mathbf{x})\boldsymbol{\nu}(\mathbf{x})\,,$$

where $\mathbf{v}_T(\mathbf{x}) = \pi(\mathbf{x})\mathbf{v}(\mathbf{x})$, $v_\nu(\mathbf{x}) = \mathbf{v}(\mathbf{x})\cdot\boldsymbol{\nu}(\mathbf{x})$.
Especially for vector $\mathbf{m}$, we have for almost every $\mathbf{x}$ in $\partial\Omega$, $\quad \mathbf{m}(\mathbf{x}) = \mathbf{m}_T(\mathbf{x}) + m_\nu(\mathbf{x})\boldsymbol{\nu}(\mathbf{x})$.
We will denote by ∂_T the tangential derivative, by ∂_ν the normal derivative and by ∇_T the tangential gradient. For every smooth enough vector field $\mathbf{v}$, we have

$$\mathrm{d}\mathbf{v} = \pi(\partial_T\mathbf{v}_T)\pi + v_\nu(\partial_T\boldsymbol{\nu}) + (\partial_\nu\mathbf{v}_T)\,{}^t\boldsymbol{\nu} + \boldsymbol{\nu}(\partial_T v_\nu - {}^t\mathbf{v}_T(\partial_T\boldsymbol{\nu}) + (\partial_\nu v_\nu)\,{}^t\boldsymbol{\nu})\,,\quad \text{on } \partial\Omega\,. \tag{22}$$

We then have

$$\epsilon(\mathbf{v}) = \epsilon_T(\mathbf{v}) + \boldsymbol{\nu}\,{}^t\epsilon_S(\mathbf{v}) + \epsilon_S(\mathbf{v})\,{}^t\boldsymbol{\nu} + \epsilon_\nu(\mathbf{v})\boldsymbol{\nu}\,{}^t\boldsymbol{\nu}\,,\quad \text{on } \partial\Omega\,, \tag{23}$$

$$\sigma(\mathbf{v}) = \sigma_T(\mathbf{v}) + \boldsymbol{\nu}\,{}^t\sigma_S(\mathbf{v}) + \sigma_S(\mathbf{v})\,{}^t\boldsymbol{\nu} + \sigma_\nu(\mathbf{v})\boldsymbol{\nu}\,{}^t\boldsymbol{\nu}\,,\quad \text{on } \partial\Omega\,, \tag{24}$$

with

$$\left|\begin{array}{l} 2\epsilon_T(\mathbf{v}) = \pi(\partial_T\mathbf{v}_T)\pi + \pi\,{}^t(\partial_T\mathbf{v}_T)\pi + 2v_\nu\partial_T\boldsymbol{\nu}\,,\\ 2\epsilon_S(\mathbf{v}) = \partial_\nu\mathbf{v}_T + \nabla_T v_\nu - (\partial_T\boldsymbol{\nu})\mathbf{v}_T\,,\\ \epsilon_\nu(\mathbf{v}) = \partial_\nu v_\nu\,,\end{array}\right. \quad\text{and}\quad \left|\begin{array}{l} \sigma_T(\mathbf{v}) = 2\mu\epsilon_T(\mathbf{v}) + \lambda(\mathrm{tr}(\epsilon_T(\mathbf{v}) + \epsilon_\nu(\mathbf{v}))I_2\,,\\ \sigma_S(\mathbf{v}) = 2\mu\epsilon_S(\mathbf{v})\,,\\ \sigma_\nu(\mathbf{v}) = 2\mu\epsilon_\nu(\mathbf{v}) + \lambda(\mathrm{tr}(\epsilon_T(\mathbf{v}) + \epsilon_\nu(\mathbf{v}))\,.\end{array}\right.$$

Remark. It can be observed that $\epsilon_T(\mathbf{v})$ and $\sigma_T(\mathbf{v})$ correspond to some symmetric $(n-1)\times(n-1)$-matrices, and that $\epsilon_S(\mathbf{v})$ and $\sigma_S(\mathbf{v})$ correspond to some vectors of dimension $n-1$, such that in some orthogonal basis $(\mathbf{t}_1,\dots,\mathbf{t}_{n-1},\boldsymbol{\nu})$, where $\mathbf{t}_1,\dots,\mathbf{t}_{n-1}$ belong to the tangent space, tensors $\epsilon(\mathbf{v})$ and $\sigma(\mathbf{v})$ are represented by matrices

$$\begin{pmatrix} \epsilon_T(\mathbf{v}) & \epsilon_S(\mathbf{v}) \\ {}^t\epsilon_S(\mathbf{v}) & \epsilon_\nu(\mathbf{v}) \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} \sigma_T(\mathbf{v}) & \sigma_S(\mathbf{v}) \\ {}^t\sigma_S(\mathbf{v}) & \sigma_\nu(\mathbf{v}) \end{pmatrix}.$$

We first estimate

$$I = \int_S^T\int_{\partial\Omega_N} 2(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}'\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,d\boldsymbol{\gamma}\,dt\,.$$

From (22), we deduce

$$\mathbf{u}'\cdot(\mathbf{m}\cdot\nabla)\mathbf{u} = \mathbf{u}_T'\cdot(\partial_T\mathbf{u}_T)\mathbf{m}_T + (u_\nu\mathbf{u}_T' - u_\nu'\mathbf{u}_T)\cdot(\partial_T\boldsymbol{\nu})\mathbf{m}_T + u_\nu'\nabla_T u_\nu\cdot\mathbf{m}_T + m_\nu(\mathbf{u}_T'\partial_\nu\mathbf{u}_T + u_\nu'\partial_\nu u_\nu)\,.$$

Using this form, we expand I and we study each term.

• Estimate of $I_1 = \displaystyle\int_S^T\int_{\partial\Omega_N} 2m_\nu u_\nu'\nabla_T u_\nu\cdot\mathbf{m}_T\,d\boldsymbol{\gamma}\,dt$.

$\partial\Omega$ is a compact manifold of dimension $n-1$. Then there exists a finite number of local maps associated to a partition of unity $(\theta_1,\dots,\theta_k)$. We denote $U_j = \mathrm{supp}(\theta_j)$. We then have

$$\int_{\partial\Omega_N} 2m_\nu u_\nu'\nabla_T u_\nu\cdot\mathbf{m}_T\,d\boldsymbol{\gamma} = \int_{\partial\Omega} 2m_\nu u_\nu'\nabla_T u_\nu\cdot\mathbf{m}_T\,d\boldsymbol{\gamma} = \sum_{j=1}^k\int_{U_j} 2m_\nu\theta_j u_\nu'\nabla_T u_\nu\cdot\mathbf{m}_T\,d\boldsymbol{\gamma}\,.$$

For $j\in\{1,\dots,k\}$, we consider the j-th term in the above sum. For simplicity, we have denoted θ_j and U_j by θ and U, respectively. We have

$$\mathbf{m}_T = \sum_{i=1}^{n-1} m^i\mathbf{a}_i\,.$$

We write $|g| = |\det(g)|$ and $W = \boldsymbol{\phi}^{-1}(U)$ and we get

$$\int_U 2m_\nu\theta u_\nu'\nabla_T u_\nu\cdot\mathbf{m}_T\,d\boldsymbol{\gamma} = \int_W 2(m_\nu\circ\boldsymbol{\phi})(\theta\circ\boldsymbol{\phi})(u_\nu'\circ\boldsymbol{\phi})\Big(\sum_{i=1}^{n-1}\frac{\partial(u_\nu\circ\boldsymbol{\phi})}{\partial\xi_i}m^i\Big)|g|^{1/2}\,d\boldsymbol{\gamma}\,. \tag{25}$$

We write $v_\nu = u_\nu \circ \boldsymbol{\phi} \in \mathrm{H}^{1/2}(W)$ and we get: $\quad \|v_\nu\|_{\mathrm{H}^{1/2}(W)} \leq C\,\|u_\nu\|_{\mathrm{H}^{1/2}(U)}$.
We now introduce a partition of W,

$$\begin{array}{l} W^+ = \{(\xi_1,\ldots,\xi_{n-1}) \in \mathbb{R}^{n-1} \,/\, m^1(\xi_1,\ldots,\xi_{n-1}) > 0\} \cap W\,,\\ W^- = \{(\xi_1,\ldots,\xi_{n-1}) \in \mathbb{R}^{n-1} \,/\, m^1(\xi_1,\ldots,\xi_{n-1}) < 0\} \cap W\,, \end{array}$$

and we look at the term with m^1 on W^+ in the right-hand side of (25).
We write $\psi = \Big((m_\nu \circ \boldsymbol{\phi})(\theta \circ \boldsymbol{\phi}) m^1 |g|^{1/2}\Big)^{1/2}$. So we have

$$\begin{aligned} \int_{W^+} 2(m_\nu \circ \boldsymbol{\phi})(\theta \circ \boldsymbol{\phi}) v_\nu' \frac{\partial v_\nu}{\partial \xi_1} m^1 |g|^{1/2}\, d\boldsymbol{\gamma} &= \int_{W^+} 2\psi^2 v_\nu' \frac{\partial v_\nu}{\partial \xi_1}\, d\boldsymbol{\gamma} \\ &= \int_{W^+} 2\psi v_\nu' \frac{\partial(\psi v_\nu)}{\partial \xi_1}\, d\boldsymbol{\gamma} - \int_{W^+} \psi (v_\nu^2)' \frac{\partial \psi}{\partial \xi_1}\, d\boldsymbol{\gamma}\,. \end{aligned} \tag{26}$$

Consider the right-hand side of (26). We have

$$\int_S^T \int_{W^+} \psi (v_\nu^2)' \frac{\partial \psi}{\partial \xi_1}\, d\boldsymbol{\gamma}\, dt = \Big[\int_{W^+} \psi v_\nu^2 \frac{\partial \psi}{\partial \xi_1}\, d\boldsymbol{\gamma}\Big]_S^T\,.$$

We have: $\quad \Big|\int_{W^+} \psi v_\nu^2 \frac{\partial \psi}{\partial \xi_1}\, d\boldsymbol{\gamma}\Big| \leq C \int_{\partial\Omega_N} |\mathbf{u}|^2\, d\boldsymbol{\gamma} \leq C \int_{\partial\Omega} |\mathbf{u}|^2\, d\boldsymbol{\gamma}$.
A classical trace result gives: $\quad \int_{\partial\Omega} |\mathbf{u}|^2\, d\boldsymbol{\gamma} \leq C \|\mathbf{u}\|^2_{\mathbb{H}^1(\Omega)}$.
Furthermore, Poincaré's inequality and Korn's inequality lead to: $\quad \|\mathbf{u}\|^2_{\mathbb{H}^1(\Omega)} \leq C\, E(\mathbf{u}, t)$.
Then, since $E(\mathbf{u}, .)$ is non-increasing, we get $C > 0$, independent of $\mathbf{u}$, such that

$$\Big|\int_S^T \int_{W^+} \psi (v_\nu^2)' \frac{\partial \psi}{\partial \xi_1}\, d\boldsymbol{\gamma}\, dt\Big| \leq C\, E(\mathbf{u}, S)\,. \tag{27}$$

Now, for the first term of (26), we define G such that

$$G = \psi v_\nu\,, \quad \text{in } W^+ \times \mathbb{R}\,, \qquad G = 0\,, \text{in } (\mathbb{R}^{n-1} \setminus W^+) \times \mathbb{R}\,.$$

We can write: $\quad \int_{W^+} 2\psi v_\nu' \frac{\partial(\psi v_\nu)}{\partial \xi_1}\, d\boldsymbol{\gamma} = \int_{\mathbb{R}^{n-1}} 2G' \frac{\partial G}{\partial \xi_1}\, d\boldsymbol{\gamma}$.
Let us denote by $\hat{G}$ the Fourier transform of G with respect to ξ_1.
We get: $\quad \int_{W^+} 2\psi v_\nu' \frac{\partial(\psi v_\nu)}{\partial \xi_1}\, d\boldsymbol{\gamma} = \int_{\mathbb{R}^{n-1}} 2i\pi\eta_1 (\hat{G}^2)'\, d\boldsymbol{\gamma}$.
By classical arguments, we get, independently of $\mathbf{u}$ and t,

$$\Big|\int_{\mathbb{R}^{n-1}} \eta_1 \hat{G}^2\, d\boldsymbol{\gamma}\Big| = O(\|G\|^2_{\mathrm{H}^{1/2}(\mathbb{R}^{n-1})}) = O(\|u_\nu\|^2_{\mathrm{H}^{1/2}(\partial\Omega_N)}) = O(\mathbf{u}\|^2_{\mathrm{H}^1(\Omega)})\,.$$

Thanks to (9), $E(\mathbf{u}, .)$ is non-increasing and we obtain

$$\Big|\int_S^T \int_{W^+} 2\psi v_\nu' \frac{\partial(\psi v_\nu)}{\partial \xi_1}\, d\boldsymbol{\gamma}\, dt\Big| \leq C\, E(\mathbf{u}, S)\,. \tag{28}$$

Then, from (27) and (28), we deduce: $\quad \Big|\int_{W^+} 2\psi^2 v_\nu' \frac{\partial v_\nu}{\partial \xi_1}\, d\boldsymbol{\gamma}\Big| \leq C\, E(\mathbf{u}, S)$.
For the integral in W^-, we replace a_1 by $-a_1$, m^1 by $-m^1$, respectively and proceed as above. We can also get similar results concerning the integral terms containing m^i, for $i \in \{2, \ldots, n-1\}$.
Finally, we get

$$|I_1| \leq C\, E(\mathbf{u}, S)\,. \tag{29}$$

• Estimate of $I_2 = \int_S^T \int_{\partial\Omega_N} 2m_\nu \mathbf{u}_T' \cdot \partial_T \mathbf{u}_T \mathbf{m}_T\, d\boldsymbol{\gamma}\, dt$.
We write $\quad \mathbf{u}_T =^t (u_T^1, \ldots, u_T^{n-1})$.

We have $\quad 2m_\nu(\mathbf{u}'_T\cdot\partial_T)\mathbf{u}_T\mathbf{m}_T=\sum_{i=1}^{n-1}2m_\nu u_T^{i'}(\nabla_T u_T^i\cdot\mathbf{m}_T)$.
As well as for I_1, we get for all $i\in\{1,\dots,n-1\}$, $\quad \Big|\int_S^T\int_{\partial\Omega_N}2m_\nu u_T^{i'}(\nabla_T u_T^i\cdot\mathbf{m}_T)\,d\boldsymbol{\gamma}\,dt\Big|\le C\,E(\mathbf{u},S)$.
And then,

$$|I_2|\le C\,E(\mathbf{u},S)\,. \tag{30}$$

• Estimate of $I_3=\displaystyle\int_S^T\int_{\partial\Omega_N}2m_\nu^2\mathbf{u}'_T\cdot\partial_\nu\mathbf{u}_T\,d\boldsymbol{\gamma}\,dt$.
We have $\quad \partial_\nu\mathbf{u}_T=2\epsilon_S(\mathbf{u})-\nabla_T\mathbf{u}_\nu+\partial_T\boldsymbol{\nu}\,\mathbf{u}_T$.
Now, for some $\theta>0$, $\quad \Big|\displaystyle\int_{\partial\Omega_N}2m_\nu^2\mathbf{u}'_T\cdot(\partial_T\boldsymbol{\nu}\,\mathbf{u}_T)\,d\boldsymbol{\gamma}\Big|\le\frac{C}{\theta}\int_{\partial\Omega_N}m_\nu|\mathbf{u}'_T|^2\,d\boldsymbol{\gamma}+\theta\int_{\partial\Omega_N}|\mathbf{u}|^2\,d\boldsymbol{\gamma}$.
Therefore, as in (27),

$$\Big|\int_{\partial\Omega_N}2m_\nu^2\mathbf{u}'_T\cdot(\partial_T\boldsymbol{\nu}\,\mathbf{u}_T)\,d\boldsymbol{\gamma}\Big|\le\frac{C}{\theta}\int_{\partial\Omega_N}m_\nu|\mathbf{u}'|^2\,d\boldsymbol{\gamma}+\theta E(\mathbf{u},t)\,. \tag{31}$$

Moreover,

$$\Big|\int_{\partial\Omega_N}4m_\nu^2\mathbf{u}'_T\cdot\epsilon_S(\mathbf{u})\,d\boldsymbol{\gamma}\Big|\le\frac{C}{\theta}\int_{\partial\Omega_N}m_\nu|\mathbf{u}'|^2\,d\boldsymbol{\gamma}+\theta\int_{\partial\Omega_N}m_\nu|\epsilon_S(\mathbf{u})|^2\,d\boldsymbol{\gamma}\,. \tag{32}$$

Let us now estimate the remaining term $\displaystyle\int_S^T\int_{\partial\Omega_N}2m_\nu^2\mathbf{u}'_T\cdot\nabla_T\mathbf{u}_\nu\,d\boldsymbol{\gamma}\,dt$. We have

$$\int_S^T\int_{\partial\Omega_N}2m_\nu^2\mathbf{u}'_T\cdot\nabla_T\mathbf{u}_\nu\,d\boldsymbol{\gamma}\,dt=\Big[\int_{\partial\Omega_N}2m_\nu^2\mathbf{u}_T\cdot\nabla_T\mathbf{u}_\nu\,d\boldsymbol{\gamma}\Big]_S^T-\int_S^T\int_{\partial\Omega_N}2m_\nu^2\mathbf{u}_T\cdot\nabla_T\mathbf{u}'_\nu\,d\boldsymbol{\gamma}\,dt\,. \tag{33}$$

We have

$$\int_{\partial\Omega_N}2m_\nu^2\mathbf{u}_T\cdot\nabla_T\mathbf{u}'_\nu\,d\boldsymbol{\gamma}=-\int_{\partial\Omega_N}2\mathbf{u}'_\nu\mathrm{div}_T(m_\nu^2\mathbf{u}_T)\,d\boldsymbol{\gamma}+\int_\Gamma 2\mathbf{u}'_\nu m_\nu^2\mathbf{u}_T\cdot\boldsymbol{\tau}\,d\boldsymbol{\gamma}=-\int_{\partial\Omega_N}2\mathbf{u}'_\nu\mathrm{div}_T(m_\nu^2\mathbf{u}_T)\,d\boldsymbol{\gamma}\,.$$

Observe that $\mathrm{div}_T(m_\nu^2\mathbf{u}_T)=m_\nu^2\mathrm{div}_T(\mathbf{u}_T)+2m_\nu\nabla_T(m_\nu)\cdot\mathbf{u}_T$. We then get

$$\Big|\int_{\partial\Omega_N}2\mathbf{u}'_\nu\mathrm{div}_T(m_\nu^2\mathbf{u}_T)\,d\boldsymbol{\gamma}\Big|\le\Big|\int_{\partial\Omega_N}2\mathbf{u}'_\nu m_\nu^2\mathrm{div}_T(\mathbf{u}_T)\,d\boldsymbol{\gamma}\Big|+\Big|\int_{\partial\Omega_N}4\mathbf{u}'_\nu m_\nu\nabla_T(m_\nu)\cdot\mathbf{u}_T\,d\boldsymbol{\gamma}\Big|\,.$$

Therefore,

$$\Big|\int_{\partial\Omega_N}2\mathbf{u}'_\nu\mathrm{div}_T(m_\nu^2\mathbf{u}_T)\,d\boldsymbol{\gamma}\Big|\le\theta\int_{\partial\Omega_N}(|\mathbf{u}|^2+m_\nu|\mathrm{div}_T(\mathbf{u}_T)|^2)\,d\boldsymbol{\gamma}+\frac{C}{\theta}\int_{\partial\Omega_N}m_\nu|\mathbf{u}'|^2\,d\boldsymbol{\gamma}\,.$$

But $|\mathrm{div}_T(\mathbf{u}_T)|^2\le 2\epsilon_T(\mathbf{u}_T):\epsilon_T(\mathbf{u}_T)$.
Therefore,

$$\Big|\int_{\partial\Omega_N}2\mathbf{u}'_\nu\mathrm{div}_T(m_\nu^2\mathbf{u}_T)\,d\boldsymbol{\gamma}\Big|\le\theta E(\mathbf{u},t)+\theta\int_{\partial\Omega_N}m_\nu\epsilon_T(\mathbf{u}_T):\epsilon_T(\mathbf{u}_T)\,d\boldsymbol{\gamma}+\frac{C}{\theta}\int_{\partial\Omega_N}m_\nu|\mathbf{u}'|^2\,d\boldsymbol{\gamma}\,. \tag{34}$$

Let us now study the remaining term in (33), $\displaystyle\Big[\int_{\partial\Omega_N}2m_\nu^2\mathbf{u}_T\cdot\nabla_T\mathbf{u}_\nu\,d\boldsymbol{\gamma}\Big]_S^T$.

$$\begin{aligned}\int_{\partial\Omega_N}2m_\nu^2\mathbf{u}_T\cdot\nabla_T\mathbf{u}_\nu\,d\boldsymbol{\gamma}&=-\int_{\partial\Omega_N}2\mathrm{div}_T(m_\nu^2\mathbf{u}_T)\mathbf{u}_\nu\,d\boldsymbol{\gamma}\\&=-\int_{\partial\Omega_N}2m_\nu^2\mathrm{div}_T(\mathbf{u}_T)\mathbf{u}_\nu\,d\boldsymbol{\gamma}-\int_{\partial\Omega_N}4m_\nu(\nabla_T m_\nu\cdot\mathbf{u}_T)\mathbf{u}_\nu\,d\boldsymbol{\gamma}\,.\end{aligned}$$

We have: $\quad \Big|\displaystyle\int_{\partial\Omega_N}4m_\nu(\nabla_T m_\nu\cdot\mathbf{u}_T)\mathbf{u}_\nu\,d\boldsymbol{\gamma}\Big|\le C\int_{\partial\Omega_N}|\mathbf{u}|^2\,d\boldsymbol{\gamma}$.
Hence,

$$\Big|\int_{\partial\Omega_N}4m_\nu(\nabla_T m_\nu\cdot\mathbf{u}_T)\mathbf{u}_\nu\,d\boldsymbol{\gamma}\Big|\le C\,E(\mathbf{u},t)\,. \tag{35}$$

It remains $\Big[\displaystyle\int_{\partial\Omega_N} 2m_\nu^2 \mathrm{div}_T(\mathbf{u}_T)\mathbf{u}_\nu \, d\boldsymbol{\gamma}\Big]_S^T$.
For a given $t>0$, let us define $\xi \in \mathrm{H}^1(\partial\Omega_N)$ such that

$$\left\{ \begin{array}{ll} \xi - \Delta_T \xi = \mathrm{div}_T(\mathbf{u}_T)(t)\,, & \text{in } \partial\Omega_N\,, \\ \xi = 0 & \text{on } \Gamma\,. \end{array} \right.$$

We have $\mathrm{div}_T(\mathbf{u}_T)(t) \in \mathrm{H}^{-1/2}(\partial\Omega_N)$, then ξ satisfies

$$\|\xi\|_{\mathrm{H}^1(\partial\Omega_N)} \le C\,\|\mathbf{u}_T\|_{L^2(\partial\Omega_N, T(\partial\Omega_N))}\,, \qquad \xi \in \mathrm{H}^{3/2}(\partial\Omega_N) \;\text{ and }\; \|\xi\|_{\mathrm{H}^{3/2}(\partial\Omega_N)} \le C\,\|\mathbf{u}_T\|_{\mathrm{H}^{1/2}(\partial\Omega_N, T(\partial\Omega_N))}\,.$$

Using ξ, we get

$$\int_{\partial\Omega_N} 2m_\nu^2 \mathrm{div}_T(\mathbf{u}_T)\mathbf{u}_\nu \, n\boldsymbol{\gamma} = \int_{\partial\Omega_N} 2m_\nu^2 \mathbf{u}_\nu \xi \, d\boldsymbol{\gamma} - \int_{\partial\Omega_N} 2m_\nu^2 \mathbf{u}_\nu \Delta_T \xi \, d\boldsymbol{\gamma}\,.$$

We may write: $\Big|\displaystyle\int_{\partial\Omega_N} 2m_\nu^2 \mathbf{u}_\nu \xi \, d\boldsymbol{\gamma}\Big| = O\Big(\int_{\partial\Omega_N} (|\mathbf{u}_\nu|^2 + |\xi|^2)\, d\boldsymbol{\gamma}\Big) = O\Big(\int_{\partial\Omega_N} |\mathbf{u}|^2 \, d\boldsymbol{\gamma}\Big)$. Then,

$$\Big|\int_{\partial\Omega_N} 2m_\nu^2 \mathbf{u}_\nu \xi \, d\boldsymbol{\gamma}\Big| \le C\,E(\mathbf{u},t)\,.$$

Let us write

$$\int_{\partial\Omega_N} 2m_\nu^2 \mathbf{u}_\nu \Delta_T \xi \, d\boldsymbol{\gamma} = \int_{\partial\Omega_N} (-\Delta_T)^{1/4}(m_\nu^2 \mathbf{u}_\nu)(-\Delta_T)^{3/4}(\xi)\, d\boldsymbol{\gamma}\,.$$

As above, we get: $\Big|\displaystyle\int_{\partial\Omega_N} 2m_\nu^2 \mathbf{u}_\nu \Delta_T \xi \, d\boldsymbol{\gamma}\Big| = O\big(\|\mathbf{u}_\nu\|_{\mathrm{H}^{1/2}(\partial\Omega_N)} \|\xi\|_{\mathrm{H}^{3/2}(\partial\Omega_N)}\big) = O\big(\|\mathbf{u}\|^2_{\mathbb{H}^1(\Omega)}\big)$. Hence,

$$\Big|\int_{\partial\Omega_N} 2m_\nu^2 \mathbf{u}_\nu \Delta_T \xi \, d\boldsymbol{\gamma}\Big| \le C\,E(\mathbf{u},t)\,.$$

We finally get

$$\Big|\Big[\int_{\partial\Omega_N} 2m_\nu^2 \mathrm{div}(\mathbf{u}_T)\mathbf{u}_\nu \, d\boldsymbol{\gamma}\Big]_S^T\Big| \le C\,E(\mathbf{u},S)\,. \tag{36}$$

With (35) and (36), we obtain

$$\Big|\Big[\int_{\partial\Omega_N} 2m_\nu^2 \mathbf{u}_T \cdot \nabla_T \mathbf{u}_\nu \, d\boldsymbol{\gamma}\Big]_S^T\Big| \le C\,E(\mathbf{u},S)\,. \tag{37}$$

Now, thanks to (31), (32), (34) and (37),

$$\begin{aligned} |I_3| \;\le\; & \theta \int_S^T \int_{\partial\Omega_N} m_\nu (|\epsilon_S(\mathbf{u})|^2 + \epsilon_T(\mathbf{u}_T) : \epsilon_T(\mathbf{u}_T))\, d\boldsymbol{\gamma}\, dt + \theta \int_S^T E(\mathbf{u},t)\, dt \\ & + \frac{C}{\theta} \int_S^T \int_{\partial\Omega_N} m_\nu |\mathbf{u}'|^2 \, d\boldsymbol{\gamma}\, dt + C\,E(\mathbf{u},S)\,. \end{aligned} \tag{38}$$

• Estimate of $I_4 = \displaystyle\int_S^T \int_{\partial\Omega_N} 2m_\nu (u_\nu \mathbf{u}'_T - u'_\nu \mathbf{u}_T) \cdot (\partial_T \boldsymbol{\nu}) \mathbf{m}_T \, d\boldsymbol{\gamma}\, dt$.

By Cauchy-Schwarz inequality, we get: $|I_4| \le \theta \displaystyle\int_S^T \int_{\partial\Omega_N} |\mathbf{u}|^2 \, d\boldsymbol{\gamma}\, dt + \frac{C}{\theta} \int_S^T \int_{\partial\Omega_N} m_\nu |\mathbf{u}'|^2 \, d\boldsymbol{\gamma}\, dt$. Then,

$$|I_4| \le \theta \int_S^T E(\mathbf{u},t)\, dt + \frac{C}{\theta} \int_S^T \int_{\partial\Omega_N} m_\nu |\mathbf{u}'|^2 \, d\boldsymbol{\gamma}\, dt\,. \tag{39}$$

• Estimate of $I_5 = \displaystyle\int_S^T \int_{\partial\Omega_N} 2m_\nu^2 u'_\nu \partial_\nu u_\nu \, d\boldsymbol{\gamma}\, dt$.

Similarly, we get

$$|I_5| \le \theta \int_S^T \int_{\partial\Omega_N} m_\nu |\partial_\nu \mathbf{u}_\nu|^2 \, d\boldsymbol{\gamma}\, dt + \frac{C}{\theta} \int_S^T \int_{\partial\Omega_N} m_\nu |\mathbf{u}'|^2 \, d\boldsymbol{\gamma}\, dt\,. \tag{40}$$

• End of the proof.
We now use (29), (30), (38), (39) and (40) and, for some $\theta > 0$, we get

$$\begin{aligned} |I| \quad \leq \quad & \theta \int_S^T \int_{\partial\Omega_N} m_\nu (|\epsilon_S(\mathbf{u})|^2 + \epsilon_T(\mathbf{u}_T) : \epsilon_T(\mathbf{u}_T) + |\partial_\nu u_\nu|^2) \, d\boldsymbol{\gamma} \, dt \\ & + \theta \int_S^T E(\mathbf{u}, t) \, dt + \frac{C}{\theta} \int_S^T \int_{\partial\Omega_N} m_\nu |\mathbf{u}'|^2 \, d\boldsymbol{\gamma} \, dt + CE(\mathbf{u}, S) \, . \end{aligned} \tag{41}$$

Now, using (23) and (24), we get

$$\sigma(\mathbf{u}) : \epsilon(\mathbf{u}) \geq 2\mu (|\epsilon_S(\mathbf{u})|^2 + \epsilon_T(\mathbf{u}_T) : \epsilon_T(\mathbf{u}_T) + |\partial_\nu u_\nu|^2) \, .$$

Therefore, for every $\theta > 0$ small enough,

$$\int_S^T \int_{\partial\Omega_N} m_\nu \big(-\sigma(\mathbf{u}) : \epsilon(\mathbf{u}) + \theta (|\epsilon_S(\mathbf{u})|^2 + \epsilon_T(\mathbf{u}_T) : \epsilon_T(\mathbf{u}_T) + |\partial_\nu u_\nu|^2) \big) \, d\boldsymbol{\gamma} \, dt \leq 0 \, .$$

This completes the proof. □

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